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RINGS OF NORMAL AND NONNORMAL NUMBERS

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To the memory of Gerold Wagner

ABSTRACT

We construct an uncountable ring of real numbers all of whose nonzero elements are normal to base b and nonnormal to base ab, where a, b are any pair of integers greater than one with $(a,b)=1$.

For any prime a and any integer $b \geq 2$ with $(a,b)=1$, G. Wagner [10] gave an explicit example of an uncountable ring of real numbers all of whose nonzero elements are normal to base b and nonnormal to base ab. In this paper, we construct a new example of a ring of the same properties with a not necessarily prime (see Theorem 5). Our construction is based on an algebraic independence result for the special values of certain gap series (in [8]) and a sufficient condition of the normality to base b and the nonnormality to base ab of the numbers of the form $\Sigma_{n\geq 1} A_n a^{-\lambda_n} b^{-\mu_n}$, where λ_n $(\geq 1), \mu_n$ (≥ 1), and A_n are integers for all $n \geq 1$ (see Theorem 3). We also give in Theorem 1 precise discrepancy estimates from above as well as from below for numbers of the form $\Sigma_{n\geq 1} a^{-\lambda_n} b^{-\mu_n}$, whose normality to base b has been studied by Korobov (cf. [3], [4], [5]).

Let $b > 1$ be an integer. A real number α is said to be normal to base b, if the sequence $\{\alpha b^n\}_{n\geq 1}$ is uniformly distributed mod 1, namely if

$$\lim_{N\to\infty} N^{-1} D([u,v); \alpha b^n, 0 \leq n < N) = 0$$

for any $[u,v) \subset [0,1)$, or what amounts to the same thing (cf. [6]), if

$$\lim_{N\to\infty} N^{-1} D(\alpha b^n, 0 \leq n < N) = 0,$$

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where as usual

$$D(\alpha b^n, 0 \le n < N) = \sup_{0 \le u < v \le 1} D([u,v); \alpha b^n, 0 \le n < N),$$

$$D([u,v); \alpha b^n, 0 \le n < N) = |A([u,v); \alpha b^n, 0 \le n < N) - (v-u)N|,$$

and $A([u,v); \alpha b, 0 \le n < N)$ is the number of integers n with $0 \le n < N$ for which $u \le \alpha b^n - [\alpha b^n] < v$. Here $[x]$ is the integral part of x.

Let $0.a_1 a_2 \cdots = a_1 b^{-1} + a_2 b^{-2} + \cdots$ be the b-adic expansion of $\alpha - [\alpha]$. For any $b_1 \cdots b_\ell \in \{0, 1, \ldots, b-1\}^\ell$, let $A(\alpha, b_1 \cdots b_\ell, N)$ denote the number of n with $1 \le n \le N$ for which $a_n a_{n+1} \cdots a_{n+\ell-1} = b_1 b_2 \cdots b_\ell$ and let

$$I(b_1 \cdots b_\ell) = \left[\frac{b_1}{b} + \frac{b_2}{b^2} + \cdots + \frac{b_\ell}{b^\ell}, \frac{b_1}{b} + \frac{b_2}{b^2} + \cdots + \frac{b_\ell + 1}{b^\ell}\right).$$

Then we have

$$D(I(b_1 \cdots b_\ell); \alpha b^n, 0 \le n < N) = |A(\alpha, b_1 \cdots b_\ell, N) - b^{-\ell} N|,$$

and hence α is normal to base b if and only if

$$\lim_{N \to \infty} N^{-1} |A(\alpha, b_1 \cdots b_\ell, N) - b^{-\ell} N| = 0$$

for any integer $\ell \ge 1$ and any $b_1 \cdots b_\ell \in \{0, 1, \ldots, b-1\}^\ell$.

THEOREM 1: *Let $a, b > 1$ be integers with $(a,b) = 1$, let $\{\lambda_n\}_{n \ge 1}$ and $\{\mu_n\}_{n \ge 1}$ be sequences of positive integers strictly increasing for all sufficiently large n, and let*

$$\alpha = \sum_{n=1}^{\infty} \frac{1}{a^{\lambda_n} b^{\mu_n}}. \tag{1}$$

Then there is a positive constant c_0 such that

$$D(\alpha b^x, 0 \le x < N) < c_0 \left(\sum_{k=1}^{n-1} \frac{\mu_{k+1} - \mu_k}{a^{\lambda_k}} + \frac{N - \mu_n}{a^{\lambda_n}} + \lambda_n^2 \sqrt{a}^{\lambda_n} \right), \tag{2}$$

for all N, where n is defined by $\mu_n < N \le \mu_{n+1}$.

Furthermore, there are positive constants c_1 and c_2 such that for any $\ell > c_1$ and $b_1 \cdots b_\ell \in \{0, 1, \ldots, b-1\}^\ell$ we have

$$D(I(b_1 \cdots b_\ell); \alpha b^x, 0 \le x < N) > \frac{c_2}{b^\ell} \frac{\mu_{n+1} - \mu_n}{a^{\lambda_n}} \tag{3}$$

for all $n \ge 1$ and for some integer N with $\mu_n < N \le \mu_{n+1}$.

COROLLARY 1: *If*

$$\lim_{n\to\infty} \lambda_n^2 \sqrt{a}^{\lambda_n}/\mu_n = 0,$$

the number α defined by (1) *is normal to base b.*

To prove Theorem 1 we need the following lemmas.

LEMMA 1: (Erdös and Turan cf. [6]). *There exists an absolute constant c such that*

$$D(x_1,\ldots,x_N) \le c\left(\frac{N}{Q} + \sum_{q-1}^{Q}\frac{1}{q}|\sum_{n=1}^{N} e^{2\pi i q x_n}|\right)$$

for any real numbers $x_1,\ldots,x_n$ and any positive integer Q.

For any integer $b, m > 1$, let $\tau(b,m)$ denote the order of b mod m.

LEMMA 2: ([3], Lemma 1). *Let $b > 1$ be an integer and let $p_1,\ldots,p_s$ be s distinct primes with $(b, p_1\cdots p_s) = 1$. Then there are integers $e_i = e_i(b, p_1,\ldots,p_s) > 0$ $(1 \le i \le s)$ such that*

$$\tau(b, p_1^{n_1}\cdots p_s^{n_s}) = p_1^{n_1-e_1}\cdots p_s^{n_s-e_s}\tau(b, p_1^{e_1}\cdots p_s^{e_s})$$

for all integers $n_i \ge e_i$ $(1 \le i \le s)$.

LEMMA 3: ([3], Theorem 2). *Make the same assumptions as in Lemma 2 and assume that $n_i > e_i$ and $p_i^{n_i-e_i} \nmid A$ for some i. Then we have*

$$\sum_{x=0}^{\tau(b,m)-1} e^{2\pi i A b^x/m} = 0,$$

where $m = p_1^{n_1}\cdots p_s^{n_s}$.

LEMMA 4: ([4], Lemma 2). *Make the same assumptions as in Lemma 3 and let σ be an integer with $0 \le \sigma < \tau(b,m)$. Then we have*

$$\sum_{x=0}^{\sigma} e^{2\pi i A b^x/m} < \sqrt{m}\log m.$$

Korobov [5] proved that, if $\{\lambda_n\}_{n\ge1}$ and $\{\mu_n\}_{n\ge1}$ are increasing sequences of positive integers satisfying $\mu_n \ge a^{\lambda_n}$ $(n \ge 1)$, the number α defined by (1) is normal to base b. His proof is to show that

$$|A(\alpha, b_1\cdots b_\ell, N) - b^{-\ell}N| = o(N),$$

by using Theorems 1 and 3 in [4], which were deduced from Lemmas 3 and 4 written above. We estimate $D(\alpha b^x, 0 \le x < N)$ in (2) using directly Lemmas 3 and 4 via Lemma 1.

For any prime p and an integer n, let $v_p(n)$ denote the exponent d for which $p^d | n$ and $p^{d+1} \nmid n$.

Proof of Theorem 1: We first prove (2). Let N be a large integer and let n be as in the theorem. Then we have

$$(4) \qquad D(N) \le \mu_1 + \sum_{k=1}^{n-1} D(\mu_k, \mu_{k+1}) + D(\mu_n, N),$$

where $D(m,n) := D(\alpha b^x, m \le x < n)$ and $D(N) = D(0,N)$. We put $\tau_k = \tau(b, a^{\lambda_k})$ and define integers ν_k and σ_k $(1 \le k \le n)$ by

$$\nu_k \tau_k < \mu_{k+1} - \mu_k \le (\nu_k + 1)_{\tau_k} \quad (1 \le k < n), \quad \nu_n \tau_n < N - \mu_n \le (\nu_n + 1)\tau_n,$$

$$\sigma_k = \mu_{k+1} - \mu_k - \nu_k \tau_k, \quad \sigma_n = N - \mu_n - \nu_n \tau_n,$$

so that by Lemma 2

$$(5) \qquad \sigma_k \le \tau_k \leqq a^{\lambda_k} \ll \tau_k \quad (1 \le k \le n),$$

and

$$(6) \qquad \nu_k \ll (\mu_{k+1} - \mu_k) a^{-\lambda_k} \quad (1 \le k \le n), \quad \nu_n \ll (N - \mu_n) a^{-\lambda_n}.$$

Then we have

$$(7) \ D(\mu_k, \mu_{k+1}) \le \sum_{\nu=0}^{\nu_k - 1} D(\mu_k + \nu\tau_k, \mu_k + (\nu+1)\tau_k) + D(\mu_k + \nu_k\tau_k, \mu_k + \nu_k\tau_k + \sigma_k),$$

$$D(\mu_n, N) \le \sum_{\nu=0}^{\nu_n - 1} D(\mu_n + \nu\tau_n, \mu_n + (\nu+1)\tau_n) + D(\mu_n + \nu_n\tau_n, \mu_n + \nu_n\tau_n + \sigma_n).$$

Now it follows from Lemma 1 that

$$(8) \qquad D(\mu_k + \nu_k\tau_k, \mu_k + \nu_k\tau_k + \sigma_k) \ll \frac{\sigma_k}{Q} + \sum_{q=1}^{Q} \frac{1}{q} \Big| \sum_{x=0}^{\sigma_k - 1} e(q_\alpha b^{\mu_k + \nu_k\tau_k + x}) \Big|,$$

where $e(x) = e^{2\pi i x}$. We choose Q as

$$Q = a^{\lambda_k - \ell_0}, \tag{9}$$

where $\ell_0 \in \mathbb{N}$ is a large constant. Writing

$$\alpha_k = \sum_{j=1}^{k} \frac{1}{a^{\lambda_j} b^{\mu_j}} = \frac{B_k}{a^{\lambda_k} b^{\mu_k}},$$

we have

$$\sum_{x=0}^{\sigma_k - 1} e(q\alpha b^{\mu_k + \nu_k \tau_k + x})| = |\sum_{x=0}^{\sigma_k - 1} e(q\alpha_k b^{\mu_k + \nu_k \tau_k + x})| + O(1), \tag{10}$$

noticing that $|e(u) - e(v)| \ll |u - v|$ and $\alpha - \alpha_k \ll a^{-\lambda_{k+1}} b^{-\mu_{k+1}}$.

To apply Lemma 4, we put

$$A = qB_k b^{\nu_k \tau_k} \quad \text{and} \quad m = a^{\lambda_k}, \tag{11}$$

so that $q\alpha_k b^{\mu_k + \nu_k \tau_k + x} = Ab^x/m$ in (10). Then, since $(a, b) = (a, B_n) = 1$ and $q \le Q$, there is a prime $p | a$ such that

$$v_p(A) = v_p(q) \le v_p(m) - \ell_0.$$

Thus we can apply Lemma 4 and get

$$\left| \sum_{x=0}^{\sigma_k - 1} e(q\alpha_k b^{\mu_k + \nu_k \tau_k + x}) \right| \ll \lambda_k \sqrt{a}^{\lambda_k}, \tag{12}$$

which together with (5), (8), (9), and (10) yield

$$D(\mu_k + \nu_k \tau_k, \mu_k + \nu_k \tau_k + \sigma_k) \ll \lambda_k^2 \sqrt{a}^{\lambda_k} \quad (1 \le k \le n).$$

Using Lemma 3 in place of Lemma 4, we find

$$D(\mu_k + \nu \tau_k, \mu_k + (\nu + 1)\tau_k) \ll 1 \quad (0 \le \nu < \nu_k, 1 \le k \le n).$$

Substituting these inequalities as well as (6) into (7) and (4), we obtain (2).

Proof of (3): Let n be sufficiently large. Then we have $[\alpha b^{\mu_{n+1}}] = [\alpha_n b^{\mu_{n+1}}]$, and so $[\alpha b^{j+\ell}] = [\alpha_n b^{j+\ell}]$ for any integers ℓ and j with $\ell \ge 1$ and $0 \le j \le \mu_{n+1} - \ell$.

This implies that both of the fractional parts $\{\alpha b^j\}$ and $\{\alpha_n b^j\}$ belong to the same interval $[m_j b^{-\ell}, (m_j+1)b^{-\ell})$ for some integer m_j with $0 \le m_j < b^\ell$. But $\{\alpha_n b^j\} = \{\alpha_n b^{j+\tau_n}\}$ for any $j \ge \mu_n$. Hence, if $\mu_n \le j \le \mu_{n+1} - \ell$, $\{\alpha b^j\}$ and $\{\alpha b^{j+\tau_n}\}$ belong to the same interval $[m_j b^{-\ell}, (m_j+1)b^{-\ell})$. Therefore, for any given $b_1 \cdots b_\ell \in \{0, 1, \ldots, b-1\}^\ell$, we have

$$d(\Delta; \mu_n + \nu\tau_n, \mu_n + (\nu+1)\tau_n) = d(\Delta; \mu_n, \mu_n + \tau_n),$$

where $\Delta = I(b_1 \cdots b_\ell)$ and

$$d(\Delta; m, n) = \#\{m \le x < n \mid \alpha b^x \in \Delta\} - (n-m)b^{-\ell}\},$$

so that

$$d(\Delta; 0, \mu_n + \nu\tau_n) = d(\Delta; 0, \mu_n) + \nu d(\Delta; \mu_n, \mu_n + \tau_n) \tag{13}$$

for any integer ν with $0 \le \nu \le \rho_n$, where

$$\rho_n = [(\mu_{n+1} - \mu_n - \ell)/\tau_n].$$

Now, since $D(\Delta; \mu_n, \mu_n + \tau_n)$ differs from τ_n/b^ℓ by an integer, and since $b^\ell \nmid \tau_n$ for all integers $\ell > c_\ell = c_\ell(a)$, we have

$$D(\Delta; \mu_n, \mu_n + \tau_n) = |d(\Delta; \mu_n, \mu_n + \tau_n)| \ge b^{-\ell} \tag{14}$$

for all $\ell > c_1$, where $D(\Delta; m, n) = D(\Delta; \alpha b^x, m \le x < n)$. Hence it follows from (13), (14), and Lemma 2 that

$$\begin{aligned} \max_{1 \le \nu \le \rho_n} d(\Delta; 0, \mu_n + \nu\tau_n) - \min_{1 \le \nu \le \rho_n} d(\Delta; 0, \mu_n + \nu\tau_n) &\ge (\rho_n - 1) D(\Delta; \mu_n, \mu_n + \tau_n) \\ &\ge \frac{c_2}{b^\ell} \frac{\mu_{n+1} - \mu_n}{a^{\lambda_n}} \end{aligned}$$

for infinitely many n, provided $\ell > c_1$, since we may assume that $(\mu_{n+1} - \mu_n)/a^{\lambda_n}$ is unbounded; where c_2 is independent of ℓ. Therefore, for any fixed $b_1 \cdots b_\ell$ with $\ell > c_1$ and for infinitely many n, there is an integer $\kappa_n = \kappa_n(b_1 \cdots b_\ell)$ with $1 < \kappa_n \leqq \rho_n$ such that

$$D(\Delta; 0, \mu_n + \kappa_n\tau_n) \ge \frac{c_2}{2b^\ell} \frac{\mu_{n+1} - \mu_n}{a^{\lambda_n}}.$$

Putting $N = \mu_n + \kappa_n\tau_n$, we obtain (3). This completes the proof of Theorem 1. ∎

THEOREM 2: *If*

$$\varlimsup_{n\to\infty} \mu_n/\mu_{n-1} > 1 + (\log a)/\log b,$$

then the number α defined by (1) is nonnormal to base ab.

LEMMA 5: *Let $r > 1$ be an integer and let α be real. For any $q \in \{0, 1, \dots, r-1\}$, let $L(\alpha, r, q; N)$ be the maximum length of runs of q appearing in the first N digits of the r-adic expansion of $\alpha - [\alpha]$. If there is a constant $c > 0$ such that $L(\alpha, r, q; N) > cN$ for infinitely many N, then α is nonnormal to base r.*

The proof of Lemma 5 is clear.

Proof of Theorem 2: We write $\alpha = \sum_{n\geq 1} a^{\mu_n-\lambda_n}/(ab)^{\mu_n}$. The number of digits in the ab-adic expansion of $a^{\mu_n-\lambda_n}$ is $[(\mu_n - \lambda_n)\log_{ab} a] + 1 =: \kappa_n$, say. Since $\mu_{n+1} - \kappa_{n+1} \geqq \mu_n - \kappa_n$ for all large n by the growth condition of μ_n, we have

$$L(\alpha, ab, 0; \mu_n - \kappa_n) \geq \mu_n - \kappa_n - \mu_{n-1}$$

for all large n, and so

$$L(\alpha, ab, 0; \mu_n - \kappa_n) > \frac{\rho}{2}(\mu_n - \kappa_n)$$

for infinitely many n, where

$$\rho = 1 - (1 + (\log a)/\log b) \varliminf_{n\to\infty} \mu_{n-1}/\mu_n > 0;$$

and the theorem follows from Lemma 5. ∎

The bounds (2) and (3) in Theorem 1 are implicit as functions of N; however, they give precise estimates for most of the cases in which $\{\lambda_n\}_{n\geq 1}$ and $\{\mu_n\}_{n\geq 1}$ are given explicitly. In the following Example 1, $b_1 \cdots b_\ell \in \{0, 1, \dots, b-1\}^\ell$ is any block of length $\ell > c_1$.

Example 1: The number

$$\sum_{n=1}^{\infty} \frac{1}{a^n b^{[a^{\theta n}]}}$$

is normal to base b and nonnormal to base ab if $\theta > \frac{1}{2}$. In particular, if $\theta > \frac{3}{2}$, we have

$$D(\alpha b^x, 0 \leq x < N) < c_0 N^{1-1/\theta} \tag{15}$$

for all N and

$$D(I(b_1 \cdots b_\ell); \alpha b^x, 0 \le x < N) > c_2 b^{-\ell} N^{1-1/\theta} \tag{16}$$

for infinitely many integer N. In particular, the number

$$\alpha = \sum_{n=1}^{\infty} \frac{1}{a^n b^{c^n}}$$

is normal to base b and nonnormal to base ab, where c is an integer with $c > \sqrt{a}$. Furthermore, the inequalities (15) and (16) hold with $\theta = \log c / \log a$, if $c > \sqrt{a}^3$. We note that α is a non-Liouville transcendental number (cf. [9]). (A proof of transcendency of α when $c = 2$ can be found in [7].)

The following theorem will be used in the proofs of Theorems 4 and 5.

THEOREM 3: *Let $a, b > 1$ be integers with $(a, b) = 1$, let $\{\lambda_n\}_{n\ge1}$ and $\{\mu_n\}_{n\ge1}$ be sequences of positive integers which are increasing and*

$$\mu_n \ge a^{\lambda_n}$$

for all large n, and let $\{A_n\}_{n\ge1}$ be a sequence of integers such that

$$|A_n| < a^{\lambda_n - \lambda_{n-1}}$$

for all large n and $A_n \neq 0$ for infinitely many n. Then the number

$$\alpha = \sum_{n=1}^{\infty} \frac{A_n}{a^{\lambda_n} b^{\mu_n}}$$

is normal to base b and nonnormal to base ab.

Proof: We may assume without loss of generality that $A_n \neq 0$ for all $n \ge 1$. The proof of normality is much the same as that of (2) in Theorem 1. Indeed, it is valid until (10), if we put

$$\alpha_k = \sum_{j=1}^{k} \frac{A_j}{a^{\lambda_j} b^{\mu_j}} = \frac{B_k}{a^{\lambda_k} b^{\mu_k}}$$

with $B_k = a^{\lambda_k - \lambda_{k-1}} b^{\mu_k - \mu_{k-1}} B_{k-1} + A_k$ and choose Q as

$$Q = 2^{\lambda_{k-1} - \ell_0}$$

in place of (9), so that for any prime $p|a$ we have $v_p(q) \le \lambda_{k-1} - \ell_0$ $(1 \le q \le Q)$. On the other hand, there is a prime $p_0|a$ such that $v_{p_0}(B_k) = v_{p_0}(A_k) \le v_{p_0}(a^{\lambda_k - \lambda_{k-1}})$, and so

$$v_{p_0}(A) = v_{p_0}(q) + v_{p_0}(B_k) < v_{p_0}(m) - \ell_0,$$

when A and m are defined by (11). Hence we can apply Lemma 4 and get (12). Therefore

$$D(\mu_k + \nu_k\tau_k, \mu_k + \nu_k\tau_k + \sigma_k) \ll a^{\lambda_k} 2^{-\lambda_{k-1}} + \lambda_k^2 \sqrt{a}^{\lambda_k} \quad (1 \le k \le n).$$

Similarly we have

$$D(\mu_k + \nu\tau_k, \mu_k + (\nu+1)\tau_k) \ll a^{\lambda_k} 2^{-\lambda_{k-1}} \quad (1 \le k \le n).$$

Thus we obtain

$$D(N) \ll \sum_{k=2}^{n-1} \frac{\mu_{k+1} - \mu_k}{2^{\lambda_{k-1}}} + \frac{N - \mu_n}{2^{\lambda_{n-1}}} + \lambda_n^2 \sqrt{a}^{\lambda_n} = o(N),$$

and the normality to base b is proved. The nonnormality to base ab can be proved similarly to that of Theorem 2.

G. Wagner [10] proved the following theorem:

Let p be a prime and $g \ge 2$ be an integer with $(p,g) = 1$, let $\{\lambda_n\}_{n\ge1}$ and $\{\mu_n\}_{n\ge1}$ be increasing sequences of positive integers such that

$$\lim_{n\to\infty} \lambda_n/(n\mu_{n-1}) = \infty$$

and

$$\lim_{n\to\infty} (\log \mu_n)/\lambda_n = \infty,$$

and let R be the ring generated by the set of all numbers of the form

$$\prod_{n=1}^{\infty} \left(1 + \frac{\epsilon_n}{p^{\lambda_n} q^{\mu_n}}\right), \quad \{\epsilon_n\}_{n\ge1} \in \{-1,1\}^{\mathrm{N}}.$$

Then R is uncountable and any $\alpha \in R \setminus \{0\}$ is normal to base g and nonnormal to base pg.

The construction of a ring in Theorem 5 is simpler than that of Wagner, however the proof of the theorem seems to be easier.

THEOREM 4: *Let $a, b > 1$ be integers with $(a,b) = 1$, let $\{\lambda_n\}_{n\geq 1}$ and $\{\mu_n\}_{n\geq 1}$ be sequences of integers increasing for all large n such that*

$$\lim_{n\to\infty} \lambda_n/\mu_{n-1} = \infty \tag{17}$$

and

$$\lim_{n\to\infty} (\log \mu_n)/\lambda_n = \infty, \tag{18}$$

and let R be the ring generated by the set of all numbers of the form

$$\omega = \sum_{n=1}^{\infty} \frac{\epsilon_n}{a^{\lambda_n} b^{\mu_n}}, \quad \{\epsilon_n\}_{n\geq 1} \in \{0,1\}^{\mathbf{N}}. \tag{19}$$

Then R is uncountable, $R \cap \mathbf{Q} = \{A/(ab)^n | A, n \in \mathbf{Z}\}$, and any $\alpha \in R \setminus \mathbf{Q}$ is normal to base b and nonnormal to base ab.

To construct a ring of normal numbers containing no rationals other than zero, we use the following

LEMMA 6: (Special case of Theorem in [8]). *Let $\{\lambda_{n,\gamma}\}_{n\geq 1}$ and $\{\mu_{n,\gamma}\}_{n\geq 1}$ $(\gamma \in \Gamma)$ be two families of sequences of positive integers with a parameter set Γ such that, for any finite subset of Γ suitably indexed as $\{\gamma_1, \ldots, \gamma_d\}$, the sequences $\{\lambda_\nu\}_{\nu\geq 1}$ and $\{\mu_\nu\}_{\nu\geq 1}$ defined by $\lambda_\nu = \lambda_{n,\gamma_i}$ and $\mu_\nu = \mu_{n,\gamma_i}$, where $i = i(\nu)$ is the integer i in $1 \leqq i \leqq d$ with $i \equiv \nu (\mathrm{mod}\, d)$, satisfy the conditions (17) and (18), and let*

$$f_\gamma(z) = \sum_{n=1}^{\infty} a^{-\lambda_{n,\gamma}} z^{\mu_{n,\gamma}} \quad (\gamma \in \Gamma).$$

Then the numbers $\{f_\gamma(\alpha) \,|\, \gamma \in \Gamma\}$ are algebraically independent over $\mathbf{Q}$ for any algebraic α with $0 < |\alpha| < 1$.

THEOREM 5: *Let $a, b > 1$ be integers with $(a,b) = 1$, let $\{\lambda_{n,\gamma}\}_{n\geq 1}$ and $\{\mu_{n,\gamma}\}_{n\geq 1} (\gamma \in \Gamma)$ be two families of sequences of positive integers given in Lemma 6, and let R be the ring generated by the set of all numbers of the form*

$$\omega_\gamma = \sum_{n=1}^{\infty} \frac{1}{a^{\lambda_{n,\gamma}} b^{\mu_{n,\gamma}}}, \quad \gamma \in \Gamma.$$

Then any $\alpha \in R \setminus \{0\}$ is normal to base b and nonnormal to base ab.

Example 2: Families of sequences $\{\lambda_{n,\gamma}\}_{n\geq 1}$ and $\{\mu_{n,\gamma}\}_{n\geq 1}$ $(\gamma \in \Gamma)$ satisfying the conditions in Lemma 6 can be easily found. For instance, put for brevity $g_1(x) = \log_a x$ and $g_n(x) = g_1(g_{n-1}(x))\,(n \geq 2)$, and let $\Gamma = (0,1)$. Define, for each $\gamma \in \Gamma$ and $n \geq 1$, $\lambda'_{n,\gamma}$ and $\mu'_{n,\gamma}$ by $\lambda_{1,\gamma} = 1$,

$$g_{[n^2+\gamma n]}(\lambda'_{n,\gamma}) = 1 \quad \text{and} \quad g_{[n^2+\gamma n+2]}(\mu'_{n,\gamma}) = \gamma \quad (n \geq 1),$$

and put $\lambda_{n,\gamma} = [\lambda'_{n,\gamma}]$ and $\mu_{n,\gamma} = [\mu'_{n,\gamma}]$. We remark that, for any fixed $\gamma \in \Gamma$, the sequences $\{\lambda_{n,\gamma}\}_{n\geq 1}$ and $\{\mu_{n,\gamma}\}_{n\geq 1}$ satisfy all the conditions in Theorem 4.

Remark 1: It is easily seen that $\omega_\gamma \neq \omega'_\gamma$, if $\gamma \neq \gamma'$, so that Γ, $\{\omega_\gamma | \gamma \in \Gamma\}$, and R have the same cardinality. So Example 2 yields an uncountable ring R. Putting $\alpha = 1/b$ in Lemma 6, we see that the numbers $\{\omega_\gamma | \gamma \in \Gamma\}$ are algebraically independent over $\mathbb{Q}$. In particular, the ring R in Theorem 5 contains no rationals other than zero.

Remark 2: Additive groups of normal numbers having prescribed Hausdorff dimension $\alpha\,(0 \leq \alpha < 1)$ have been constructed by many authors (see, e.g., [1], [2]). All the rings of normal numbers mentioned in this paper are of Hausdorff dimension zero, since any irrational numbers contained in these rings are Liouvillian and the set of all Liouville numbers is of Hausdorff dimension zero.

Proof of Theorem 4: We shall only prove the normality, since the other statements can be easily seen. Any $\alpha \in R$ can be written as

$$\alpha = P(\omega_1, \ldots, \omega_d), \tag{20}$$

where $P(x_1, \ldots, x_d) \in \mathbb{Z}[x_1, \ldots, x_d]$ and ω_i is defined by (19) with some $\{\varepsilon_n\}_{n\geq 1} = \{\varepsilon_{n,i}\}_{n\geq 1} \in \{0,1\}^{\mathbb{N}} (1 \leq i \leq d)$. Let $s_1, \ldots, s_d$ be positive integers with $s_1 + \cdots + s_d \leq \deg P = S$. Then we have

$$\omega_i^{s_i} = \sum_{n=1}^{\infty} W_n(s_i), \tag{21}$$

where

$$W_n(s_i) = \sum_{\substack{0 \leq \sigma_k \leq s_i,\, \sigma_n \neq 0 \\ \sigma_1 + \cdots + \sigma_n = s_i}} \frac{A(s; n; \sigma_1, \ldots, \sigma_n)}{a^{\sum_{k=1}^{n} \sigma_k \lambda_k} b^{\sum_{k=1}^{n} \sigma_k \mu_k}} \tag{22}$$

with $|A(s_i;n;\sigma_1,\dots,\sigma_n)| \ll 1$. (Here and in what follows all constants implied by the symbol $\ll$ may depend on the polynomial P.) Hence it follows that

$$\begin{aligned}\omega_1^{s_1}\cdots\omega_d^{s_d} &= \sum_{n=1}^{\infty} \sum_{\substack{1\le n_i\le n\\ \max n_i=n}} W_{n_1}(s_1)\cdots W_{n_d}(s_d)\\ &= \sum_{n=1}^{\infty} \sum_{\substack{0\le\sigma_k\le s\\ \sigma_n\neq 0}} \frac{B(n;\sigma_1,\dots,\sigma_n)}{a^{\sum_{k=1}^{n}\sigma_k\lambda_k} b^{\sum_{k=1}^{n}\sigma_k\mu_k}}\end{aligned}$$

with $|B(n;\sigma_1,\dots,\sigma_n)| \ll n^d$. Therefore we obtain

$$\begin{aligned}\alpha &= \sum_{n=1}^{\infty} \sum_{\substack{0\le\sigma_k\le S\\ \sigma_n\neq 0}} \frac{C(n;\sigma_1,\dots,\sigma_n)}{a^{\sum_{k=1}^{n}\sigma_k\lambda_k} b^{\sum_{k=1}^{n}\sigma_k\mu_k}}\\ &= \sum_{n=1}^{\infty}\sum_{\sigma=1}^{S} \sum_{0\le\sigma_k\le S} \frac{C(n;\sigma_1,\dots,\sigma_n)}{a^{\sigma\lambda_n+\sum_{k=1}^{n-1}\sigma_k\lambda_k} b^{\sigma\mu_n+\sum_{k=1}^{n-1}\sigma_k\mu_k}}\\ &= \sum_{n=1}^{\infty}\sum_{\sigma=1}^{S} \frac{C(n,\sigma)}{a^{\sigma\lambda_n+S\sum_{k=1}^{n-1}\lambda_k} b^{\sigma\mu_n+S\sum_{k=1}^{n-1}\mu_k}},\end{aligned}$$

where

$$|C(n,\sigma)| \ll n^d S^n a^{S\sum_{k=1}^{n-1}\lambda_k} b^{S\sum_{k=1}^{n-1}\mu_k},$$

so that, by (17) and (18),

$$\log|C(n,\sigma)| \ll \mu_{n-1} \tag{23}$$

for all large n.

For each (n,σ) with $n \ge 1$ and $1 \le \sigma \le S$, we put $\nu = S(n-1)+\sigma$ and set $A_\nu = C(n,\sigma)$, and

$$\Lambda_\nu = \sigma\lambda_n + S\sum_{k=1}^{n-1}\lambda_k, \quad M_\nu = \sigma\mu_n + S\sum_{k=1}^{n-1}\mu_k,$$

so that $\Lambda_\nu - \Lambda_{\nu-1} \ge \lambda_n$ and $M_\nu - M_{\nu-1} \ge \mu_n$. Then we obtain

$$\alpha = \sum_{\nu=1}^{\infty} \frac{A_\nu}{a^{\Lambda_\nu} b^{M_\nu}},$$

where, by (17), (18), and (23),

$$a^{\Lambda_\nu} < a^{(\sigma+1)\lambda_n} < \mu_n \le M_\nu \quad \text{and} \quad |A_\nu| < a^{\lambda_n} \leqq a^{\Lambda_\nu-\Lambda_{\nu-1}}$$

for all large n, and therefore the normality follows from Theorem 3. This completes the proof of Theorem 4. ∎

Proof of Theorem 5: We give the proof of normality, which is much the same as the preceding one. Any $\alpha \in R$ can be written as (20) with $\omega_i = \omega_{\gamma_i}$ for some suitably indexed $\gamma_i\,(1 \le i \le d)$, for which we may assume the conditions in Lemma 6. Then, for any $s_1, \ldots, s_d$ as above, we have (21) with

$$w_n(s_i) = \sum_{\substack{0 \le \sigma_k \le s_i, \sigma_n \ne 0 \\ \sigma_1 + \cdots + \sigma_n = s_i}} \frac{A(s_i, n; \sigma_1, \ldots, \sigma_n)}{a^{\sum_{k=1}^n \sigma_k \lambda_{k,i}} b^{\sum_{k=1}^n \sigma_k \mu_{k,i}}}$$

in place of (22), where $\lambda_{n,i} = \lambda_{n,\gamma_i}, \mu_{n,i} = \mu_{n,\gamma_i}$, and $0 \le A(s_i, n; \sigma_1, \ldots, \sigma_n) \ll 1$, and hence

$$\omega_1^{s_1} \cdots \omega_d^{d_s} = \sum_{n=1}^{\infty} \sum_{i=1}^{d} \sum_{\sigma_i=1}^{S} \frac{B(n,i,\sigma_i)}{a^{\sigma_i \lambda_{n,i} + S \sum_{(k,j)<(n,i)} \lambda_{k,j}} b^{\sigma_i \mu_{n,i} + S \sum_{(k,j)<(n,i)} \mu_{k,i}}},$$

where $\log B(n,i,\sigma_i) \ll \max\{\mu_{k,j} | (k,j) < (n,i)\}$ and $(k,j) < (n,i)$ is the lexicographic ordering. Therefore we obtain

$$\alpha = \sum_{n=1}^{\infty} \sum_{i=1}^{d} \sum_{\sigma_i=1}^{S} \frac{C(n,i,\sigma_i)}{a^{\sigma_i \lambda_{n,i} + S \sum_{(k,j)<(n,i)} \lambda_{k,j}} b^{\sigma_i \mu_{n,i} + S \sum_{(k,j)<(n,i)} \mu_{k,j}}}$$

with $|C(n,i,\sigma_i)| \ll B(n,i,\sigma_i)$.

For each (n,i,σ) with $n \ge 1$, $1 \le i \le d$, and $1 \le \sigma \le S$, we put $\nu = Sd(n-1) + S(i-1) + \sigma$ and set $A_\nu = C(n,i,\sigma_i)$ and

$$\Lambda_\nu = \sigma_i \lambda_{n,i} + S \sum_{(k,j)<(n,i)} \lambda_{k,j}, \quad M_\nu = \sigma_i \mu_{n,i} + S \sum_{(k,j)<(n,i)} \mu_{k,j}.$$

Then we have

$$\alpha = \sum_{\nu=1}^{\infty} \frac{A_\nu}{a^{\Lambda_\nu} b^{M_\nu}},$$

where A_ν, Λ_ν, and M_ν $(\nu \ge 1)$ satisfy all the conditions in Theorem 3, and the normality follows. The completes the proof of Theorem 5. ∎

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